Jackson 4.5(a)

Consider $E_i(\vec{x})$ of $\vec{E}(\vec{x})$

(summation on indices implied) $E_i(\vec{x}) = E_i|_0 + x_j(\partial_j E_i|_0) + \frac{1}{2}x_j x_k(\partial_k \partial_j E_i|_0) + \cdots$

$$\vec{F} = \int \vec{E}(\vec{x}')\rho(\vec{x}')\,d^3x'$$

$$F_i = E_i|_0 \int \rho(\vec{x}')\,d^3x' + (\partial_j E_i|_0)\int x'_j \rho(\vec{x}')\,d^3x'$$

$$+ \frac{1}{2}(\partial_k \partial_j E_i|_0)\int x'_j x'_k \rho(\vec{x}')\,d^3x' + \cdots$$

$$E_i|_0 \int \rho(\vec{x}')\,d^3x' = q E_i|_0$$

$$(\partial_j E_i|_0)\int x'_j \rho(\vec{x}')\,d^3x' = (\partial_j E_i|_0)p_j$$
$$= (\partial_i(-\partial_j \Phi))|_0\, p_j$$
$$= (\partial_i E_j|_0)p_j$$

$$\frac{1}{2}(\partial_k \partial_j E_i|_0)\int x'_j x'_k \rho(\vec{x}')\,d^3x' = \frac{1}{6}(\partial_i \partial_k E_j|_0)\int(3x'_j x'_k - r'^2\delta_{jk})\,d^3x'$$

$$= \frac{1}{6}(\partial_i \partial_k E_j|_0)Q_{jk}$$

$$\Rightarrow F_i = qE_i|_0 + p_j(\partial_i E_j|_0) + \frac{1}{6}(\partial_i \partial_k E_j|_0)Q_{jk} + \cdots$$

$$\Rightarrow \vec{F} = q\vec{E}|_0 + \left\{\vec{\nabla}[\vec{p}\cdot\vec{E}]\right\}_0 + \left\{\vec{\nabla}\left[\frac{1}{6}(\partial_k E_j)Q_{jk}\right]\right\}_0 + \cdots$$

The last equation requires justification of

$$\vec{\nabla}\vec{p}=0, \qquad \vec{\nabla}Q_{jk}=0.$$

This is because $\vec{p}$, Q_{jk} are simply not functions of the position variable $\vec{x}$, whereas Φ, $\vec{E}$ take on different values for $\vec{x}_1$, $\vec{x}_2$, in general.

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Jackson

4.5 (b) Again, with summation on indices implied,

$$N_1 = \int (x_2 F_3 - x_3 F_2)\, d^3x .$$

$$= \int \Big\{ x_2 \big[E_3|_0 + x_j (\partial_j E_3|_0) + \cdots \big)$$

$$- x_3 \big[E_2|_0 + x_j (\partial_j E_2|_0) + \cdots \big) \Big\}\, d^3x\, \rho(\vec{x})$$

$$= E_3|_0 \int x_2\, \rho(\vec{x})\, d^3x - E_2|_0 \int x_3\, \rho(\vec{x})\, d^3x$$

$$+ \partial_j E_3|_0 \left\{ \int x_2 x_j \rho(\vec{x})\, d^3x \right\} - \partial_j E_2|_0 \int x_3 x_j \rho(\vec{x})\, d^3x$$

$$+ \cdots$$

$$= \left[\vec{p} \times \vec{E}|_0 \right]_1 + \frac{1}{3} \left\{ \left(\partial_j E_3|_0 \right) Q_{2j} - \left(\partial_j E_2|_0 \right) Q_{3j} \right\} + \cdots$$

It's clear that $\partial_j E_3 = -\partial_j \partial_3 \Phi = -\partial_3 \partial_j \Phi = \partial_3 E_j$, so we have

$$N_1 = \left[\vec{p} \times \vec{E}|_0 \right]_1 + \frac{1}{3} \left\{ (\partial_3 E_j|_0) Q_{2j} - (\partial_2 E_j|_0) Q_{3j} \right\} + \cdots$$

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